Two-dimensional flow at high subsonic speeds past wedges in channels with parallel walls

By J. B. HELLIWELL

Department of Mathematics, The Royal College of Science and Technology, Glasgow

(*Received* 6 *September* 1957)

SUMMARY

Investigations are made of the plane flow of an inviscid isentropic gas at high subsonic and sonic speeds past a finite wedge of small angle set at zero incidence in a channel with parallel walls. Hodograph methods are applied to the determination of the stream function, which, under the usual transonic approximation, is a solution of Tricomi's equation. In § 2 a brief summary is given of the derivation of the fundamental solution of this equation in terms of Bessel functions.

Two models of the flow pattern are discussed. The model of Cole is examined in § 3 with the sonic line from the shoulder extending completely across the channel at right angles to the wall. In § 4 the Helliwell–Mackie model is taken in which there is a free stream breakaway at sonic velocity from the shoulder of the wedge and the velocity far downstream may be either uniformly subsonic or sonic. Values are obtained for the drag coefficient in both cases and a high degree of both qualitative and quantitative agreement is found. On the basis of the free streamline theory explicit formulae are given relating, in terms of the channel width and upstream Mach number, the drag of the wedge in the channel to that of the same wedge in the free stream.

Attention is drawn in § 5 to certain properties of plane jet flows that may be deduced from the investigation of the flow with free stream breakaway past a wedge in a channel.

1. INTRODUCTION

The two-dimensional steady flow of an inviscid isentropic gas at high subsonic and sonic speeds past a finite wedge profile in a free stream has been investigated by many workers, employing various methods based upon the hodograph transformation by which the resulting differential equation for the stream function Ψ is made linear. A common procedure, which will be followed in the present paper, is to use the so-called transonic approximation in which this differential equation becomes the well-known Tricomi equation

$$\frac{\partial^2\Psi}{\partial u^2}+u\frac{\partial^2\Psi}{\partial v^2}=0,$$

where $U = a^*(1-u/(\gamma+1))$ and $V = a^*v/(\gamma+1)$ are the cartesian components of velocity, the axes being taken parallel and perpendicular to the uniform flow at infinity, a^* is the velocity of the gas when the Mach number is 1 and γ is the adiabatic index of the gas. In this approximation $V = 0$ at infinity upstream of the wedge and u, v are small. Henceforth the suffix 1 will refer to conditions at infinity upstream. Furthermore, it is known that Ψ is proportional to the y-coordinate.

It should be remarked* that recent work by Vincenti, Wagoner & Fisher (1956) shows that even more accuracy may be expected from the transonic approximation if solutions of Tricomi's equation

$$\frac{\partial^2 \Psi'}{\partial u^2} + u \frac{\partial^2 \Psi'}{\partial v^2} = 0$$

are used in which u is interpreted in a somewhat different manner from that in the present paper, and Ψ' is the stream function Ψ multiplied by a certain other function of u.

Guderley & Yoshihara (1950), Cole (1951) and Helliwell & Mackie (1957) have obtained solutions of Tricomi's equation under different hypotheses regarding the flow pattern downstream of the shoulder of the wedge. In each model the flow, accelerating from zero velocity at the tip of the wedge, attains sonic velocity at the shoulder. The solution of Guderley & Yoshihara is of considerable complexity since it includes a determination of the flow field between the sonic line and the limiting characteristic of the Prandtl–Meyer expansion at the shoulder. Less difficult analysis is involved in Cole's solution in which the supersonic region degenerates into a sonic line straight and normal to the direction of the flow far upstream; the flow pattern is investigated upstream of this line. In the work of Helliwell & Mackie the sonic line is specified as a free streamline starting from the shoulder and the velocity is subsonic or sonic throughout the entire field of flow. A comparison of these various models shows a close similarity in the pressure distribution over the wedge nose upstream of the shoulder and hence correspondingly small variations in the estimate of the drag coefficient of the wedge.

An extension of the approach of Guderley & Yoshihara to the situation when the wedge is placed symmetrically in a wind tunnel has been carried out recently by Marschner (1956). However, he only investigates the case of the choked tunnel. In the present paper similar investigations are made of the flow past a wedge at zero incidence in a two-dimensional channel with parallel walls under more general conditions of downstream flow. In § 3 the basic model of Cole is used with the sonic line extending from the shoulder of the wedge normally to the channel wall. In § 4 the Helliwell–Mackie model is taken with the sonic line appearing as a free streamline from the shoulder. Expressions are obtained for the drag coefficient in each case and the channel walls are found to produce similar

* The author is indebted to a referee for drawing his attention to these results.

effects upon its value, as compared with that in a free stream, for both models of the flow. In the free streamline model, equation (25) taken in conjunction with figure 8 gives correction terms which yield explicitly, in terms of the upstream Mach number and the channel width, the relation between the drag of the wedge in the channel and of the same wedge in a free stream with the same Mach number. When using these results it should be ensured that the channel width has been taken sufficiently large for the flow far downstream not to become supersonic; the limiting relationship between channel width and upstream Mach number for uniform sonic velocity downstream is shown in figure 6.

Finally in § 5 attention is drawn to a result concerning jet flows that may be deduced from the investigations of the flows with free streamlines past the wedge in a channel.

2. Flow patterns in a subsonic stream

In the subsequent analysis we shall be concerned with flow nowhere supersonic. In this section we summarize briefly the equations relating the variables which define the flow patterns. These relationships only hold, of course, within the limits of the transonic approximation.

The dimensionless velocity variables u and v have already been defined. The Mach number M is given by $1-M^2 = u$. The polar angle θ of the velocity is related to v by $v = (\gamma+1)\theta$. The local pressure coefficient is defined by

$$c_p = (p-p_1)/(\tfrac{1}{2}\rho_1 U_1^2),$$

where p is the pressure and ρ is the density. It is straightforward to show that, according to the linearized transonic theory,

$$c_p = 2(u-u_1)/(\gamma+1). \tag{1}$$

The equation of continuity and the condition for irrotational flow are respectively

$$uu_x - v_y = 0, \qquad u_y + v_x = 0,$$

which may be inverted to yield the hodograph equations

$$uy_v - x_u = 0, \qquad x_v + y_u = 0. \tag{2}$$

If x is eliminated these lead to the equation of Tricomi

$$y_{uu} + uy_{vv} = 0,$$

in which $u > 0$ represents subsonic flow. If we set

$$r = \tfrac{2}{3}u^{3/2} = \tfrac{2}{3}(1-M^2)^{3/2}, \tag{3}$$

then, on solving Tricomi's equation by the method of separation of variables, simple solutions are obtained of the type

$$y = r^{1/3}e^{\pm\lambda v}\mathscr{C}_{\pm 1/3}(\lambda r), \tag{4}$$

where $\mathscr{C}_{\pm 1/3}(\lambda r)$ is any linear combination of Bessel functions of the order indicated and λ is any constant, real or imaginary.

3. Model 1. Sonic line normal to channel wall

In this section we employ the model of Cole for flow at near sonic velocity past a wedge profile. The basic feature of this model is that the sonic line from the shoulder is straight and normal to the flow at infinity upstream. It is pointed out by Cole in his original paper that, so far as the flow over the front portion of the wedge is concerned, it is a good approximation to replace the actual shape of the sonic line in this way, and hence a good estimate should be obtained for the drag coefficient. At low subsonic Mach numbers the supersonic region is small and so the approximation is close to being correct; for larger Mach numbers, whilst the actual sonic line is by no means straight away from the wedge, yet the upstream influence of the errors in the supersonic region introduced by the approximation is weaker.

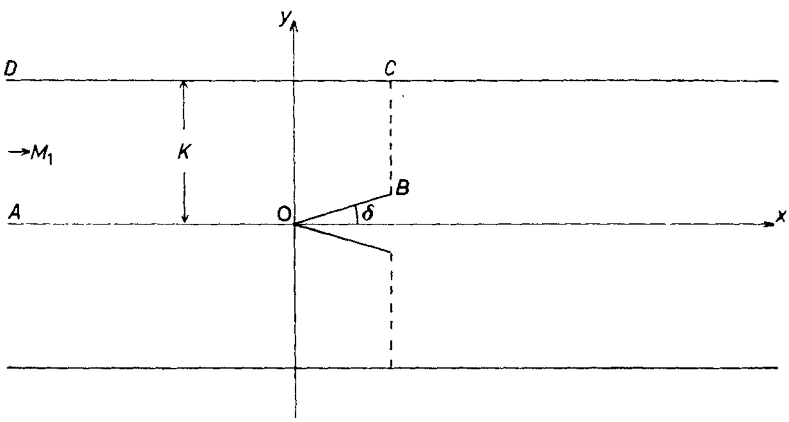

Figure 1. Model 1. Physical plane.

Consider then a wedge of semi-angle δ placed symmetrically in a channel of semi-width K as shown in figure 1. The x-axis is taken as the axis of symmetry and the y-axis passes through the tip of the wedge. Hence, we need only consider the flow in the upper half-plane $y \geqslant 0$. The channel is supposed to be blocked by the sonic line which extends normally from the channel wall to the shoulder of the wedge. This is represented by BC. AOB is the dividing streamline $\Psi = 0$ on which at A, the point at infinity upstream, the velocity is subsonic with Mach number M_1, at O, the wedge tip, the velocity is zero and along the wedge face OB the flow accelerates to attain sonic velocity at B, the shoulder of the wedge. DC is the streamline $\Psi = k$ with associated $y = K$ corresponding to the channel wall along which the flow accelerates steadily from its subsonic upstream velocity to sonic speed at C. Along the wedge face OB we also have $v = v_0 = (\gamma + 1)\delta$.

The boundary value problem may now be established in the hodograph plane, as shown in figure 2. CD is the line $y = K$. AOB is the line $y = 0$. BC is the sonic line $x =$ constant, from which it follows, since $u = 0$, that

$x_v = 0$, whence using (2) we have $y_r = 0$. Two other lines of constant y are shown. Clearly there is a finite singularity at the point DA.

The problem is now stated in terms of r and v. The condition along BC becomes $r^{1/3}y_r = 0$ when $r = 0$ for all $0 \leqslant v \leqslant v_0$ which is automatically satisfied provided that y_r remains bounded at $r = 0$. The simple solutions (4) show that we must take $y_r = 0$ at $r = 0$ as the condition to fit this requirement. Hence we have

$$\begin{aligned} y &= K, & v &= 0, & 0 &\leqslant r < r_1, \\ y &= 0, & v &= 0, & r &> r_1, \\ y &= 0, & v &= v_0, & r &\geqslant 0, \\ y_r &= 0, & 0 &\leqslant v \leqslant v_0, & r &= 0. \end{aligned}$$

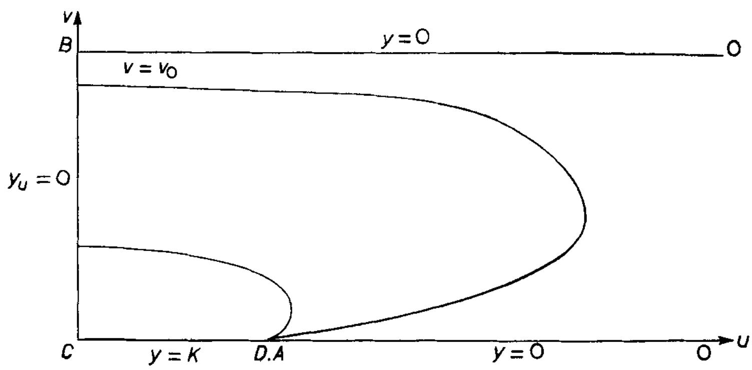

Figure 2. Model 1. Hodograph plane.

The linearization principle, applied to the stagnation condition at the tip O, leads to

$$x = 0, \qquad y = 0, \qquad \text{as } r \to \infty.$$

Finally, taking the wedge to be of unit length,

$$x = 1 \quad \text{at } v = v_0, \qquad r = 0.$$

A solution of the problem is obtained by taking a combination of the simple solutions (4) in the form

$$y = \int_0^\infty f(\lambda) r^{1/3} J_{-1/3}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda. \tag{5}$$

Differentiating under the sign of integration, we have

$$y_r = -\int_0^\infty \lambda f(\lambda) r^{1/3} J_{2/3}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} d\lambda.$$

Here $f(\lambda)$ is an arbitrary function of λ that is to be determined so that (5) satisfies the first two boundary conditions, since it is immediately apparent that the last two are satisfied automatically. Hence $f(\lambda)$ is given by

$$\int_0^\infty f(\lambda) r^{1/3} J_{-1/3}(\lambda r)\,d\lambda = K, \qquad 0 \leqslant r < r_1,$$
$$= 0, \qquad r > r_1.$$

An application of the Hankel inversion theorem and a simple integration then yields the result that

$$f(\lambda) = \lambda \int_0^{r_1} K r^{2/3} J_{-1/3}(\lambda r)\,dr = K r_1^{2/3} J_{2/3}(\lambda r_1).$$

Inserting this in (5), the solution of the problem becomes

$$y = K r_1^{2/3} r^{1/3} \int_0^\infty J_{2/3}(\lambda r_1) J_{-1/3}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0}\,d\lambda. \tag{6}$$

That this form has a singularity of the correct type at $r = r_1$ may be confirmed in the usual way. It may also be noted that (6) simplifies into Cole's solution for the wedge in the sonic free stream if $K r_1^{4/3}$ tends to a finite constant as $r_1 \to 0$ so that the channel becomes infinitely wide as the upstream Mach number tends to unity. This behaviour of K is confirmed in a later result (equation (9)).

The x-coordinate is now obtained from equation (2). After a little analysis we find that

$$x = 1 - (\tfrac{3}{2})^{1/3} K r^{2/3} r_1^{2/3} \int_0^\infty J_{2/3}(\lambda r_1) J_{2/3}(\lambda r) \frac{\cosh \lambda(v_0 - v)}{\sinh \lambda v_0}\,d\lambda. \tag{7}$$

The unit additive constant in this expression arises from the condition that the sonic line on which $r = 0$ is $x = 1$ when the wedge is taken to have unit length.

In the subsequent work a series representation for the x-coordinate on the face of the wedge will be useful. To obtain this we set $v = v_0$ in (7) and expand $\operatorname{cosech} \lambda v_0$ in partial fractions by

$$\operatorname{cosech} \lambda v_0 = \frac{1}{\lambda v_0} + 2 \sum_{n=1}^\infty \frac{(-1)^n \lambda v_0}{(\lambda v_0)^2 + (n\pi)^2}.$$

After some manipulation and use of standard results concerning integrals involving Bessel functions, we ultimately derive the series

$$x = 1 - \frac{(\frac{3}{2})^{1/3} K r^{2/3} r_1^{2/3}}{v_0} \left\{ \frac{3}{4}\left(\frac{r}{r_1}\right)^{2/3} + 2 \sum_{n=1}^\infty (-1)^n K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) I_{2/3}\left(\frac{n\pi r}{v_0}\right) \right\}, \qquad 0 < r \leqslant r_1,$$

$$= 1 - \frac{(\frac{3}{2})^{1/3} K r^{2/3} r_1^{2/3}}{v_0} \left\{ \frac{3}{4}\left(\frac{r_1}{r}\right)^{2/3} + 2 \sum_{n=1}^\infty (-1)^n I_{2/3}\left(\frac{n\pi r_1}{v_0}\right) K_{2/3}\left(\frac{n\pi r}{v_0}\right) \right\}, \qquad r \geqslant r_1. \tag{8}$$

Because of the asymptotic behaviour of $K_\nu(z)$ for large z, the stagnation condition, $x = 0$ as $r \to \infty$, applied to the latter representation gives

$$Kr_1^{4/3} = 2(\tfrac{2}{3})^{4/3} v_0. \tag{9}$$

Writing r_1 in terms of M_1 through (3) and replacing v_0 by $(\gamma+1)\delta$, we thus have the result.

$$\{\delta(\gamma+1)\}^{1/3} K = 2[(1-M_1^2)\{\delta(\gamma+1)\}^{-2/3}]^{-2}.$$

This is the relation between the semi-channel width and the upstream Mach number that, on account of continuity of the flow, holds for the present model, if the channel is of such a width that sonic velocity is attained everywhere across a section through the wedge shoulder.

The drag coefficient is defined in the customary way as

$$C_D = D/(\tfrac{1}{2}\rho_1 U_1^2),$$

where D is the drag on the upper face of the wedge and, according to the linearized theory, is given by

$$D = \delta \int C_p \, dx,$$

the integrations being taken over the length of the wedge. Introducing the relations (1) and (3) it can then be shown that

$$C_D = -2\frac{(\frac{3}{2})^{2/3}\delta}{\gamma+1}\left\{\int_0^\infty r^{2/3}\left(\frac{\partial x}{\partial r}\right)_{v=v_0} dr + r_1^{2/3}\right\}. \tag{10}$$

The series representations (8) may now be substituted into (10) and after considerable algebra, in which (9) is used to replace K in terms of r_1, an expression for C_D in the form of an infinite series is obtained as

$$\frac{(\gamma+1)^{1/3}}{\delta^{5/3}} C_D = \frac{16}{3^{2/3}\Gamma(\frac{2}{3})\pi^{4/3}}\left(\frac{3r_1}{2v_0}\right)^{-2/3}\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{n^{4/3}} K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)\right\} + \\ + \frac{4\zeta(2)}{\pi^2}\left(\frac{3r_1}{2v_0}\right)^{-4/3} - \frac{2}{3}\left(\frac{3r_1}{2v_0}\right)^{2/3},$$

where $\zeta(z)$ is the zeta function of Riemann. For values of the upstream Mach number M_1 little different from unity, a series expansion in powers of $(1-M_1^2)$ may be obtained for C_D in which the leading terms are

$$C_D = C_D^* - \frac{2\delta}{\gamma+1}(1-M_1^2) + O(1-M_1^2)^3, \tag{11}$$

where C_D^* is the finite value of the drag coefficient in Cole's model for the wedge in the sonic free stream. As M_1 decreases from unity, the value of C_D for a wedge in a channel diverges from the corresponding value of C_D for an identical wedge in a free stream with the same upstream Mach number, but reference to equation (72) of Cole's paper shows that the difference is of the order $(1-M_1^2)^3$. A specific comparison is shown graphically in figure 3.

4. Model 2. Sonic line a section of a free streamline

We now turn to investigate the flow pattern past a wedge profile symmetrically placed in a channel, employing the model incorporating a free streamline proposed by Helliwell & Mackie. For the wedge in a channel the following general properties of the flow are laid down. The dividing streamline $\Psi = 0$ is straight from infinity upstream, where the velocity is subsonic with Mach number M_1, to the tip of the wedge where there is a stagnation point. The upper branch of $\Psi = 0$ then follows the upper face of the wedge, the gas accelerating to attain sonic velocity at the shoulder at which point the

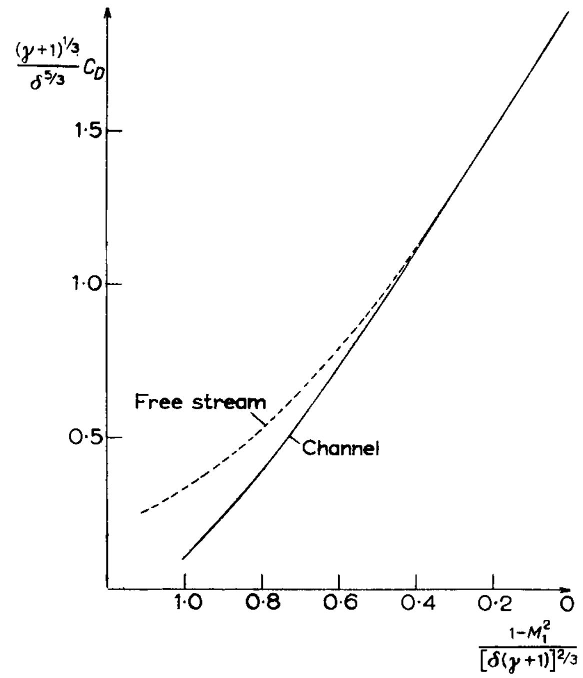

Figure 3. Model 1. Drag coefficient.

streamline breaks away from the wedge, retaining sonic velocity until it again becomes parallel to the channel walls. Thereafter it remains straight, the velocity of the gas along it meantime decelerating from sonic speed to some high subsonic velocity with Mach number M_2 ($> M_1$). The critical case of a channel with uniform sonic velocity everywhere in the flow field far downstream is merely a special case of the above with $M_2 = 1$. More generally, the model describes a flow pattern with uniform subsonic velocity upstream, a local sonic region in the neighbourhood of the shoulder of the wedge and ultimately a uniform subsonic velocity downstream in addition to the wake, regarding which certain information may be deduced. From a theoretical point of view there is the added advantage that throughout the whole flow field the velocity is nowhere supersonic and hence limit lines cannot occur to render invalid the transformation back to the physical plane.

The pattern described in the preceding paragraph is shown in figure 4. The semi-channel and semi-wake widths are K and H respectively. The upstream portion of the dividing streamline $\Psi = 0$ is taken as the line of the x-axis with origin at the tip of the wedge. The channel walls are thus the straight lines $y = \pm K$. Because of symmetry only the flow in the upper

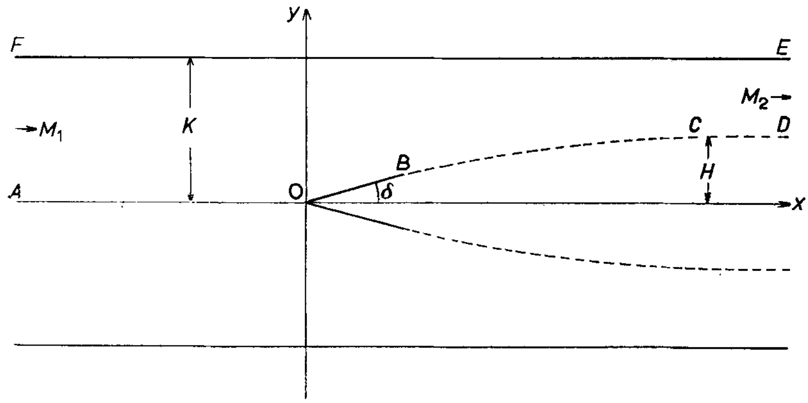

Figure 4. Model 2. Physical plane.

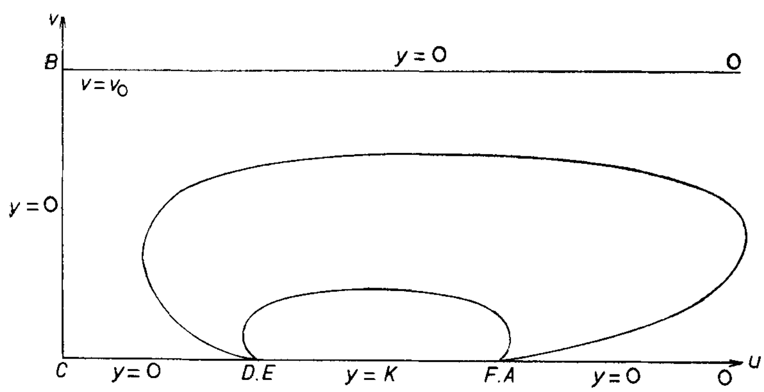

Figure 5. Model 2. Hodograph plane.

half-plane $y \geqslant 0$ need be considered. $AOBCD$, then, is the dividing streamline. At the point A far upstream the Mach number is M_1 with associated $r = r_1$; at O the velocity is zero. Along OB, the wedge face, the component $v = v_0 = (\gamma+1)\delta$, where δ is the semi-angle of the wedge; the gas accelerates to sonic velocity at B and retains this speed along the section BC. At C the velocity is again parallel to the x-axis and along CD

the gas decelerates until at D far downstream the Mach number is M_2 with associated $r = r_2$ ($< r_1$). FE is the upper channel wall along which the gas velocity varies from Mach number M_1 at F to M_2 at E.

The boundary value problem is now set up in the hodograph plane which is shown in figure 5. It is recalled that, to the order of the transonic approximation, Ψ is proportional to y. $AOBCD$ is the line $y = 0$, EF is the line $y = K$, and two other lines of constant y are shown. Finite singularities occur at the two pairs of coincident points AF and DE, corresponding to upstream and downstream conditions, respectively. The remaining boundary conditions, in terms of r and v, are

$$\begin{aligned} y &= 0, & v &= 0, & 0 &\leqslant r < r_2, \\ y &= K, & v &= 0, & r_2 &< r < r_1, \\ y &= 0, & v &= 0, & r &> r_1, \\ y &= 0, & v &= v_0, & r &\geqslant 0, \\ y &= 0, & 0 &\leqslant v \leqslant v_0, & r &= 0. \end{aligned}$$

The stagnation condition at the tip of the wedge is, according to the transonic approximation,

$$x = 0, \qquad y = 0 \quad \text{as } r \to \infty.$$

For a wedge of unit length, we also have the further condition

$$x = 1, \qquad v = v_0 \quad \text{at } r = 0.$$

The solution of the present problem may be derived from a combination of the simple solutions (4) of the type

$$y = \int_0^\infty g(\lambda) r^{1/3} J_{1/3}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} \, d\lambda.$$

Whereas this automatically satisfies the last two boundary conditions for any $g(\lambda)$, it only satisfies them all if $g(\lambda)$ is a solution of the equations

$$\begin{aligned} \int_0^\infty g(\lambda) r^{1/3} J_{1/3}(\lambda r)\, d\lambda &= 0, & 0 &\leqslant r < r_2, \\ &= K, & r_2 &< r < r_1, \\ &= 0, & r &> r_1. \end{aligned}$$

The function $g(\lambda)$ may be determined by an application of the Hankel inversion theorem and we obtain

$$g(\lambda) = K\{r_2^{2/3} J_{-2/3}(\lambda r_2) - r_1^{2/3} J_{-2/3}(\lambda r_1)\}.$$

Hence the formal solution of the problem is

$$y = K r^{1/3} \int_0^\infty \{r_2^{2/3} J_{-2/3}(\lambda r_2) - r_1^{2/3} J_{-2/3}(\lambda r_1)\} J_{1/3}(\lambda r) \frac{\sinh \lambda(v_0 - v)}{\sinh \lambda v_0} \, d\lambda. \quad (12)$$

This has singularities at $r = r_1$ and $r = r_2$ when $v = 0$. That they are of the correct type may easily be seen by an investigation of the form of y in the neighbourhood of $r = r_1$, $r = r_2$. We also note that if we let $r_2 \to r_1$ and $K \to \infty$ simultaneously, we derive the special case of the flow in a free stream. In fact, by carrying through this limiting process on y in equation (12) using the relation (14) between K, r_1 and r_2, we find that we recover the solution for flow past a wedge of unit length in a free stream given by equation (3) of the paper of Helliwell & Mackie.

The x-coordinate of the solution is next determined. A direct substitution from (12) into (2) and the resulting integrations yield

$$x = (\tfrac{3}{2})^{1/3} K r^{2/3} \int_0^\infty \{r_2^{2/3} J_{-2/3}(\lambda r_2) - r_1^{2/3} J_{-2/3}(\lambda r_1)\} J_{-2/3}(\lambda r) \frac{\cosh \lambda(v_0 - v)}{\sinh \lambda v_0} \, d\lambda. \tag{13}$$

The constant which appears as a result of the integrations may be shown to be zero as a consequence of the stagnation condition at the tip of the wedge.

The wedge will be taken to be of unit length and we have therefore $x = 1$ at the shoulder where $v = v_0$ as $r \to \infty$. Thus, from (13), we find

$$1 = \frac{(\frac{3}{2})^{1/3} 2^{2/3} K}{\Gamma(\frac{1}{3})} \int_0^\infty \frac{\lambda^{-2/3}}{\sinh \lambda v_0} \{r_2^{2/3} J_{-2/3}(\lambda r_2) - r_1^{2/3} J_{-2/3}(\lambda r_1)\} \, d\lambda.$$

On expanding the Bessel functions, setting $\lambda v_0 = t$, and interchanging orders of integration and summation, it follows that

$$1 = \frac{(\frac{3}{2})^{1/3} K}{\Gamma(\frac{1}{3}) v_0} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{3})} \left\{ r_2^{4/3} \left(\frac{r_2}{2v_0}\right)^{2n-4/3} - r_1^{4/3} \left(\frac{r_1}{2v_0}\right)^{2n-4/3} \right\} \int_0^\infty \frac{t^{2n-4/3}\, dt}{\sinh t}.$$

This is simplified by the use of a well-known integral representation of the Riemann zeta function $\zeta(z)$, giving

$$1 = \frac{2(3)^{1/3} K v_0^{1/3}}{\sqrt{\pi}\,\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \Gamma(n-\tfrac{1}{6})(2^{2n-1/3}-1)\zeta(2n-\tfrac{1}{3}) \times \\ \times \left\{ \left(\frac{r_1}{2v_0}\right)^{2n} - \left(\frac{r_2}{2v_0}\right)^{2n} \right\}. \tag{14}$$

This equation determines the relationship between upstream and downstream Mach numbers M_1 and M_2 for a wedge of unit length placed in a channel of semi-width K, where

$$r_1 = \tfrac{2}{3}(1-M_1^2)^{3/2}, \qquad r_2 = \tfrac{2}{3}(1-M_2^2)^{3/2}. \tag{15}$$

Figure 6 shows this result graphically in the important and limiting case of a channel when the velocity is uniformly sonic downstream ($r_2 = 0$). For purposes of comparison the corresponding result (9) for the model of the previous section is also shown.

In the subsequent work series representations for the coordinates will be required. We shall obtain these by means of intermediate contour integral forms. First let us investigate the y coordinate. Setting $\lambda v_0 = t$, the result (12) becomes

$$y = \frac{K\pi r^{1/3}}{v_0}\int_0^\infty \left\{ r_2^{2/3} J_{-2/3}\left(\frac{t\pi r_2}{v_0}\right) - r_1^{2/3} J_{-2/3}\left(\frac{t\pi r_1}{v_0}\right)\right\} \times$$

$$\times J_{1/3}\left(\frac{t\pi r}{v_0}\right) \frac{\sinh \pi t(1 - v/v_0)}{\sinh \pi t}\, dt. \quad (16)$$

There are three distinct cases of this integral to be discussed, depending upon the value of r.

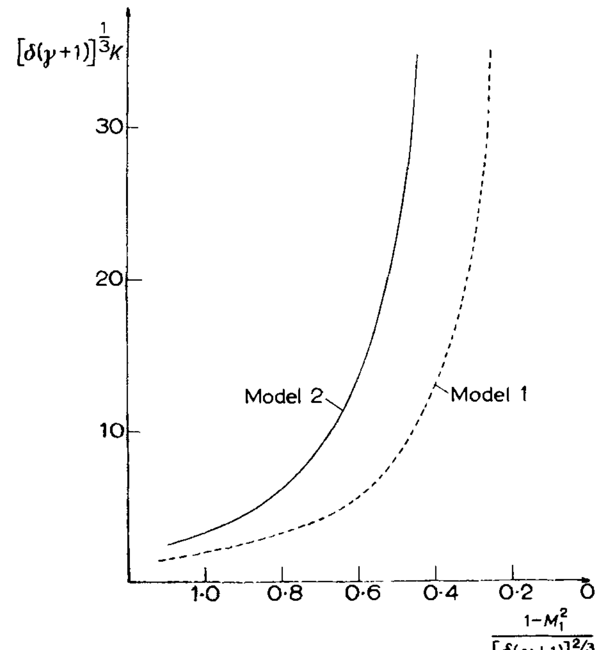

Figure 6. Upstream Mach number and critical channel width.

For $0 \leqslant r < r_2 < r_1$, consider the contour integral

$$W = -\frac{iKr^{1/3}}{v_0}\int_C \left\{ r_2^{2/3} K_{2/3}\left(\frac{\nu\pi r_2}{v_0}\right) - r_1^{2/3} K_{2/3}\left(\frac{\nu\pi r_1}{v_0}\right)\right\} I_{1/3}\left(\frac{\nu\pi r}{v_0}\right) \frac{e^{i\nu\pi(1-v/v_0)}}{\sin \nu\pi}\, d\nu,$$

where C is the contour in the complex ν plane going from $-i\infty$ to $+i\infty$, indented at the origin, and closed by the right-hand infinite semi-circle. The integrand has a branch point at the origin but is single-valued in the whole plane cut along the negative real axis. The contributions to W from the infinite semi-circle and origin indentation can be both shown to be zero by using the asymptotic and power series expansions of $K_{2/3}(z)$ and $I_{1/3}(z)$ in the respective calculations. If now we write $W = W_1 + W_2$, where W_1 is the part of the integral along the imaginary axis from 0 to $i\infty$, and set $\nu = te^{i\pi/2}$, $\nu = te^{-i\pi/2}$ in W_1 and W_2, respectively, we find after some manipulation that

$$y = \mathscr{I} W.$$

Since the poles of the integrand of W occur at the integer values of ν, the residue theorem of Cauchy applied to $\mathscr{I} W$ will yield a series form for y. We obtain

$$y = \frac{2Kr^{1/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3} K_{2/3}\left(\frac{n\pi r_2}{v_0}\right) - r_1^{2/3} K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) \right\} I_{1/3}\left(\frac{n\pi r}{v_0}\right) \sin\left(\frac{n\pi v}{v_0}\right). \quad (17)$$

In the case when $r > r_1 > r_2$, considerations similar to the above may be used to show that

$$y = \mathscr{I} \frac{iKr^{1/3}}{v_0} \int_C \left\{ r_2^{2/3} I_{-2/3}\left(\frac{\nu\pi r_2}{v_0}\right) - r_1^{2/3} I_{-2/3}\left(\frac{\nu\pi r_1}{v_0}\right) \right\} K_{1/3}\left(\frac{\nu\pi r}{v_0}\right) \frac{e^{i\nu\pi(1-v/v_0)}}{\sin \nu\pi}\, d\nu,$$

where C is the same contour as before. The Cauchy residue theorem applied to this expression gives

$$y = -\frac{2Kr^{1/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3} I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right) - r_1^{2/3} I_{-2/3}\left(\frac{n\pi r_1}{v_0}\right) \right\} K_{1/3}\left(\frac{n\pi r}{v_0}\right) \sin\left(\frac{n\pi v}{v_0}\right). \quad (18)$$

Over the range $r_2 < r < r_1$, the two forms of y already obtained suggest that we consider

$$W' = \frac{iKr^{1/3}}{v_0} \int_C \left\{ r_2^{2/3} I_{-2/3}\left(\frac{\nu\pi r_2}{v_0}\right) K_{1/3}\left(\frac{\nu\pi r}{v_0}\right) + r_1^{2/3} K_{2/3}\left(\frac{\nu\pi r_1}{v_0}\right) I_{1/3}\left(\frac{\nu\pi r}{v_0}\right) \right\} \times \\ \times \frac{e^{i\nu\pi(1-v/v_0)}}{\sin \nu\pi}\, d\nu,$$

with C the contour described previously. Once again, the contribution from the infinite semi-circle is zero, but we now find that the imaginary part of the contribution from the origin indentation is

$$-K(1 - v/v_0).$$

Hence, proceeding as before, we discover that

$$y = \mathscr{I} W' + K(1 - v/v_0),$$

and an application of the residue theorem of Cauchy yields

$$y = -\frac{2Kr^{1/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3} I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right) K_{1/3}\left(\frac{n\pi r}{v_0}\right) + \right. \\ \left. + r_1^{2/3} K_{2/3}\left(\frac{n\pi r_1}{v_0}\right) I_{1/3}\left(\frac{n\pi r}{v_0}\right) \right\} \sin\left(\frac{n\pi v}{v_0}\right) + K\left(1 - \frac{v}{v_0}\right). \quad (19)$$

The required series representations for y are given by equations (17), (18), (19). The discontinuities when $v \neq 0$ at $r = r_1$, $r = r_2$ are only apparent; continuity may be shown to exist at these points by use of a result concerning Bessel functions (Erdélyi 1953) and the Fourier series representation

$$1-\frac{v}{v_0} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi v}{v_0}, \qquad v \neq 0.$$

Of course, the value of y is still undefined at $r = r_1$, $r = r_2$ when $v = 0$.

Series representations for x may now be obtained by the use of equations (2). The constants of integration are determined by the stagnation condition at the tip of the wedge and by the necessary continuity of the x-coordinate everywhere. The three forms corresponding to equations (17), (18) and (19) are the following. For the range $0 \leqslant r < r_2 < r_1$,

$$x = \frac{2(\frac{3}{2})^{1/3}Kr^{2/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3}K_{2/3}\left(\frac{n\pi r_2}{v_0}\right) - r_1^{2/3}K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)\right\} \times$$
$$\times I_{-2/3}\left(\frac{n\pi r}{v_0}\right)\cos\left(\frac{n\pi v}{v_0}\right) + \frac{(\frac{3}{2})^{4/3}K}{2v_0}(r_1^{4/3}-r_2^{4/3}); \quad (20)$$

for $r > r_1 > r_2$,

$$x = \frac{2(\frac{3}{2})^{1/3}Kr^{2/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3}I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right) - \right.$$
$$\left. - r_1^{2/3}I_{-2/3}\left(\frac{n\pi r_1}{v_0}\right)\right\} K_{2/3}\left(\frac{n\pi r}{v_0}\right)\cos\left(\frac{n\pi v}{v_0}\right); \quad (21)$$

and for the range $r_2 < r < r_1$,

$$x = \frac{2(\frac{3}{2})^{1/3}Kr^{2/3}}{v_0} \sum_{n=1}^{\infty} \left\{ r_2^{2/3}I_{-2/3}\left(\frac{n\pi r_2}{v_0}\right)K_{2/3}\left(\frac{n\pi r}{v_0}\right) - \right.$$
$$\left. - r_1^{2/3}K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)I_{-2/3}\left(\frac{n\pi r}{v_0}\right)\right\}\cos\left(\frac{n\pi v}{v_0}\right) + \frac{(\frac{3}{2})^{4/3}K}{2v_0}(r_1^{4/3}-r_2^{4/3}). \quad (22)$$

We are now in a position to calculate the drag coefficient, C_D, which, as indicated in the previous section, is given to the order of the transonic approximation by equation (10). The evaluation is quite straightforward. The series representations (20), (21) and (22), with $v = v_0$, are inserted into the infinite integral and orders of summation and integration are interchanged. The resulting integrals involving Bessel functions are all standard types and after their evaluation a little algebra leads to the expression,

$$C_D = \tfrac{3}{2}K\delta^2\left\{\left(\frac{r_1}{v_0}\right)^2 - \left(\frac{r_2}{v_0}\right)^2\right\} - \frac{2(\frac{3}{2})^{2/3}\delta^{5/3}(r_1/v_0)^{2/3}}{(\gamma+1)^{1/3}}. \quad (23)$$

For a wedge of unit length, we express K in terms of r_1 and r_2 by equation (14) and thereby obtain

$$\frac{(\gamma+1)^{1/3}C_D}{\delta^{5/3}} = -2\left(\frac{3r_1}{2r_0}\right)^{2/3} +$$

$$+\frac{\sqrt{\pi}\,\Gamma(\tfrac{1}{3})3^{-4/3}\{(3r_1/2v_0)^2-(3r_2/2v_0)^2\}}{\sum_{n=1}^{\infty}[(-1)^{n+1}/n!]3^{-2n}\Gamma(n-\tfrac{1}{6})(2^{2n-1/3}-1)\zeta(2n-\tfrac{1}{3})\{(3r_1/2v_0)^{2n}-(3r_2/2v_0)^{2n}\}}.$$

This may be written in terms of the upstream and downstream Mach numbers M_1 and M_2. For flow sufficiently near sonic this may be expanded as a Taylor series in powers of $(1-M_1^2)$ and $(1-M_2^2)$. The leading terms of such an expansion are

$$C_D = C_D^{**} - \frac{2\delta}{\gamma+1}(1-M_1^2) + O\{(1-M_1^2)^3,\ (1-M_2^2)^3\}, \tag{24}$$

where C_D^{**} is the finite value of the drag coefficient of this model for the flow past the wedge in a sonic free stream. The result (24) should be compared

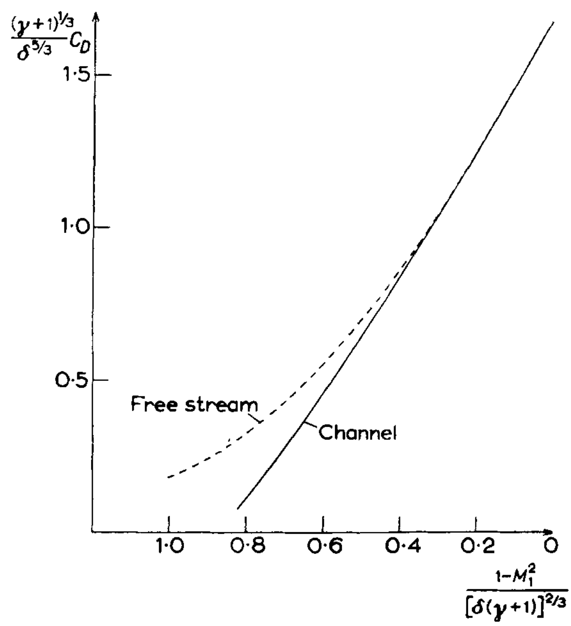

Figure 7. Model 2. Drag coefficient (critical channel).

with that for the previous model as given in equation (11). The special case of the drag coefficient for a wedge in a critical channel with variable subsonic upstream velocity and sonic velocity along the entire free streamline and uniformly far downstream is shown in figure 7. This should be compared with figure 3 for the corresponding case of the previous section. The effect of the channel walls is seen to be very similar for the two models of the flow.

A more useful form for the drag coefficient in the general case is, however, obtained if the asymptotic expansion of equation (23) is taken for large K.

The drag coefficient is then written as a series expansion in ascending powers of K^{-1} where the relation (14) has been previously used to express r_2 in terms of K and r_1. After considerable algebra we obtain an expansion of the type

$$\frac{(\gamma+1)^{1/3}}{\delta^{5/3}}\{C_{D_{\text{free}}}-C_{D_{\text{channel}}}\} = \frac{a(M_1)}{\overline{K}} + \frac{b(M_1)}{\overline{K}^2} + \frac{c(M_1)}{\overline{K}^3} + \ldots, \quad (25)$$

where $\overline{K} = K(1-M_1^2)^3/[\delta(\gamma+1)]^{5/3}$. The coefficients a, b, c, ... are extremely complicated and inelegant functions of M_1 which increase slowly as M_1 decreases from unity. The first two are shown graphically in figure 8. $C_{D_{\text{free}}}$ and $C_{D_{\text{channel}}}$ denote the drag coefficient of the same wedge in a free stream and channel, respectively, with the same upstream gas velocity. The expression (25) is thus essentially one yielding correction terms to be

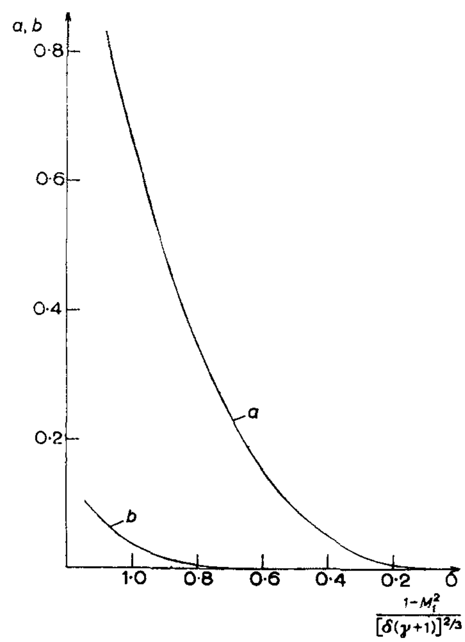

Figure 8. Model 2. Drag correction coefficients.

applied to the drag coefficient in the channel in order to obtain the corresponding value for the drag in the free stream. It is remarked that the correction to be applied is, in general, small within the range of validity of the present theory which does not apply unless the channel is wider than the length of the wedge by a factor of the order of 10. That this is so may be noted from the fact that, for a given value of M_1, the value of K yielded by figure 6 is that corresponding to uniform sonic velocity across the channel far downstream. In general, however, the velocity will be subsonic and considerations of continuity of the flow show that the value of K thus determined is a lower bound.

The location of the end of the sonic line is the point C in figure 4. Here the wake becomes parallel to the channel walls. The x-coordinate of this point is obtained by setting $r = 0$ and $v = 0$ in (20) and has the value

$$x_C = \frac{2(\frac{3}{2})^{1/3}K}{\Gamma(\frac{1}{3})v_0}\sum_{n=1}^{\infty}\left(\frac{n\pi}{2v_0}\right)^{-2/3}\left\{r_2^{2/3}K_{2/3}\left(\frac{n\pi r_2}{v_0}\right)-r_1^{2/3}K_{2/3}\left(\frac{n\pi r_1}{v_0}\right)\right\}+ \\ +\frac{(\frac{3}{2})^{4/3}K}{2v_0}(r_1^{4/3}-r_2^{4/3}). \quad (26)$$

This is finite for all r_1, $r_2 \neq 0$ so that the wake becomes parallel at a finite distance downstream of the wedge, as was laid down in the flow pattern for this model. It is of considerable interest to note that this result remains true when $r_2 = 0$, so that uniformly sonic conditions in a parallel flow would be attained at a finite distance downstream in the case when the channel width is critical.

Finally, the semi-width of the wake can be calculated. From figure 4, for a wedge of unit length, we have

$$H-\delta = \int_{y_B}^{y_C} dy = \int_{x_B}^{x_C} \theta\, dx,$$

where θ is here the slope of the sonic line. Integration by parts leads to

$$(\gamma+1)H = \int_0^{v_0} x\, dv,$$

where x is to be taken along BC. The appropriate series representation of x to be inserted in this integral is the limiting form of (20) as $r \to 0$. Thus, after the integration is performed, we find

$$H = \frac{(\frac{3}{2})^{4/3}K}{2(\gamma+1)}(r_1^{4/3}-r_2^{4/3}),$$

which by means of equations (15) may be written

$$H = K\frac{(M_2^2-M_1^2)(2-M_1^2-M_2^2)}{2(\gamma+1)}. \quad (27)$$

It should be recalled that this value is not independent of δ, as appears at first sight, since K, M_1 and M_2, are interrelated with δ through equations (14) and (15).

5. Jet flows

It is apparent that if we redraw figure 4 with FE as axis of symmetry and take the flow region bounded by the dividing streamline $AOBCD$ and its image $A'O'B'C'D'$ in FE we obtain figure 9. The physical problem portrayed is then that of a two-dimensional jet flowing from a convergent nozzle through a region of constant pressure into a duct with parallel walls. Far upstream in the channel of semi-width K, the Mach number of the flow is M_1. At the orifice BB' sonic velocity is attained at the boundary, and the gas issues as a jet with free boundaries BC, $B'C'$ along which sonic velocity is maintained. At CC' the gas enters a duct with straight, parallel walls in which the flow decelerates to a subsonic velocity with Mach number M_2 ($> M_1$).

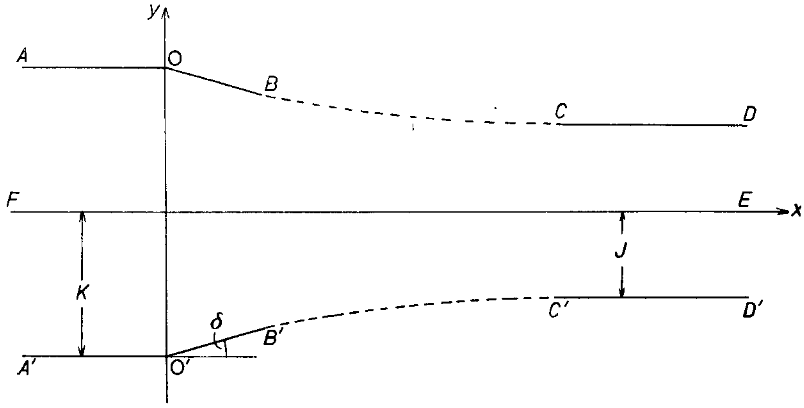

Figure 9. Physical plane for jet flow.

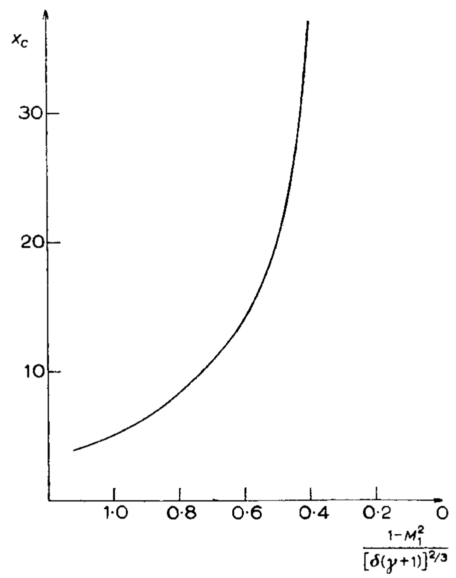

Figure 10. Downstream position of parallel sonic jet.

The case when $M_2 = 1$ is of particular interest since then the duct may be dispensed with and we have the problem of a two-dimensional jet flowing through a convergent nozzle into a region of constant pressure which has the value associated with the sonic velocity. The flow in the jet has a uniform sonic velocity far downstream, where the ultimate semi-width of the jet is J ($= K-H$). The results of the previous section in the special case when $r_2 = 0$ are thus appropriate for the solution of this problem. We take FE as x-axis and OO' as y-axis. Then, as has already been remarked, the jet becomes parallel at the point C located a finite distance downstream of the orifice. For a nozzle of unit length the value of x_C, the x-coordinate of C, is given by equation (26) with $r_2 = 0$. We recall that the relationship between the channel semi-width K and upstream mach number M_1 is shown graphically in figure 6. The variation of x_C with M_1 is shown in figure 10. Finally, it follows from equation (27) that the value of J, the semi-width of the jet, is given by

$$\frac{J}{K} = 1 - \frac{(1-M_1^2)^2}{2(\gamma+1)}.$$

As remarked at the conclusion of the previous section, it should be noted that this result is not independent of δ since K and M_1 are interrelated with this angle.

This research has been sponsored by the Air Research and Development Command U.S. Air Force under Contract AF 61(514)–1170, through the European Office A.R.D.C.

References

Cole, J. D. 1951 *J. Math. Phys.* **30**, 79.

Erdélyi, A. (Ed.) 1953 *Higher Transcendental Functions*, Vol. 2. New York: McGraw-Hill.

Guderley, G. & Yoshihara, H. 1950 *J. Aero. Sci.* **17**, 723.

Helliwell, J. B. & Mackie, A. G. 1957 *J. Fluid Mech.* **3**, 93.

Marschner, B. W. 1956 *J. Aero. Sci.* **23**, 368.

Vincenti, W. G., Wagoner, C. B. & Fisher, N. H. 1956 *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3723.